**Global Minimum Variance Portfolio**

Where:

is the vector of portfolio weights,

is the covariance matrix of asset returns,

is a vector of ones.

**Lagrangian:**

**First-Order Conditions (FOCs):**

**Solution:**

Multiply both sides of on the left by :

given ,

with

Substitute back into the original FOC:

Note: The term is a scalar, since it results from the matrix product of dimensions:

**Minimum Variance Portfolio with Target Return**

We now consider a constrained optimization problem where the investor seeks to minimize portfolio variance, subject to two linear constraints:

1.- The weights must sum to one, and

2.- The expected return of the portfolio must equal a target value R.

**Problem Formulation:**

Where:

is the vector of portfolio weights,

is the covariance matrix of asset returns,

is the vector of expected returns,

is a vector of ones,

is the target expected return of the portfolio.

**Lagrangian:**

where and are the Lagrange multiplier associated with the budget and return constraint, respectively.

First-Order Conditions (FOCs)

We compete the derivates of with respect to each variable:

With respect to :

With respect to :

**Auxiliary Quantities:**

To simplify notation, we define the following scalar quantities:

Solving the system:

From equation:

We now impose constraints by substituting this expression into each:

Multiplying both equations by 2:

Solving the linear system yield:

Substituting and back into the expression for :

This is the minimum variance portfolio that achieves a target return .